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Ultra-Multiplets: A New Representation of Rigid 2D, N = 8 Supersymmetry

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ABSTRACT

By utilizing a new procedure (the RADIO method) for deriving on-shell 2D, 2N-extended multiplets from off-shell 2D, N-extended multiplets, we derive a new on-shell 2D, N = 8 representation; the ultra-multiplet. By twisting with respect to parity, we show that many variant versions of this supermultiplet also exist.

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1 Introduction

There is little general understanding of the systematics of irreducible representations of supersymmetry. This is reflected in the fact that most of the recognized on-shell representations (i.e. the supersymmetry algebra is satisfied only with the use of the equations of motions) do not presently have off-shell representations (i.e. the supersymmetry algebra is satisfied without the use of the equations of motions). An off-shell representation of supersymmetry is obtained when a complete set of auxiliary fields is added to the on-shell theory. With the re-birth of string theory almost a decade ago [1, 2], it was possible to hope that a resolution to this problem might be forthcoming from the study of superstrings since these theories also require auxiliary fields for their complete description. Unfortunately, superstring theory has effectively contributed little to the resolution of our problem. This is not a criticism of superstring theory. Instead it is a reflection of how poor is our understanding of superstring theory.

So we are thrown back to artifice, diligence, fortune and insight to make further progress on the off-shell supersymmetric representation problem. A closely related problem is that of finding explicit irreducible representations for large values of N, the degree of "extendedness" of the supersymmetry. An important place to study these problems is within the realm of a two-dimensional space-time. This is an interesting realm in which to explore this question because solutions have consequences back on superstring theory, integrable systems and mathematical profundities. In 1988 [3], progress was made on the case of the N = 4 theory with the discovery of a minimal off-shell 2D, N = 4 supergravity multiplet. More recently there has been a clarification of the situation with N = 4 superstrings [4], where it was demonstrated that at least three off-shell 2D, N = 4 superstrings actions exist and that likely one more such theory should be possible.[3]

2 RADIO: A "Chemical" Derivation of a New Supersymmetric Representation

Before we present the explicit realization of 2D, N = 8 supersymmetry provided by the ultra-multiplet, it is useful possibly for future research to explain the method by which the ultra-multiplet was found. The genesis of our discovery is a very interesting process that we shall call "reduction, automorphic dualization, integration &

[3]Actually, we now know of a total of 7 possibly distinct N = 4 superstring actions!

oxidation" (RADIO). We will not provide a general proof of why this method works. Instead we will simply use it.

Our starting point begins with off-shell 2D, N = 4 twisted hypermultiplets (THM-I and THM-II)[4]. There are different reductions possible to 1D. Let us concentrate on the reduction of the THM-II model, where we consider a simple toroidal compactification with all fields only dependent on the τ-coordinate of the world sheet. However, we retain all of the Grassmann coordinates of the original 2D theory. This has the effect of doubling the number of supersymmetries. So we go from a 2D, N = 4 model to a 1D, N = 8 model. At this stage, equation (4) of reference [4] reads,

$$
\begin{aligned}
D_{Ii}\mathcal{T} &= (\gamma^3)_I{}^J\Psi_{Ji} \quad , \\
D_{Ii}\mathcal{X}_j{}^k &= i\left[\delta_i{}^k\Psi_{Ij} \ - \ \tfrac{1}{2}\delta_j{}^k\Psi_{Ii}\right] \quad , \\
\mathcal{X}_i{}^i &= 0 \quad , \quad \mathcal{X}_i{}^j - (\mathcal{X}_j{}^i)^* \ = \ 0 \quad , \\
D_{Ii}\Psi_{Jj} &= \tfrac{1}{2}C_{ij}C_{IJ}\bar{J} \quad , \\
D_{\alpha i}\bar{J} &= 0 \quad , \quad m - (m)^* \ = \ 0 \quad , n - (n)^* \ = \ 0 \quad , \\
\bar{D}^{Ii}\Psi_{Jj} &= i\delta_j{}^i(\gamma^3\gamma^0)_J{}^I\left(\partial_\tau\mathcal{T}\right) \ + \ 2(\gamma^0)_J{}^I\left(\partial_{\tau a}\mathcal{X}_j{}^i\right) \\
&\qquad + \ i\tfrac{1}{2}\delta_J{}^I\delta_j{}^i m \ + \ \tfrac{1}{2}(\gamma^3)_J{}^I\delta_j{}^i n \quad . \\
D_{Ii}J &= i4C_{ij}(\gamma^0)_{IJ}\left(\partial_\tau\bar{\Psi}^{Jj}\right) \quad , \\
D_{Ii}n &= -i2(\gamma^3\gamma^0)_I{}^J\left(\partial_\tau\Psi_{Ji}\right) \quad , \\
D_{Ii}m &= -2(\gamma^0)_I{}^J\left(\partial_\tau\Psi_{Ji}\right) \quad ,
\end{aligned} \tag{1}
$$

where we have replaced all spinor indices (α, β,...) by internal symmetry indices (i.e. I, J,...) to emphasize their no longer being related to spin.

Next we perform a transformation that we call a "1D automorphic duality transformation" [5]. Our investigation within the realm of 1D supersymmetric representations seem to indicate that auxiliary fields can be avoided entirely in 1D. This is a very unusual "transform" that formally replaces the would-be "auxiliary fields" of a 1D supermultiplet by propagating fields. What this amounts to is replacing every would-be "auxiliary field" by the τ-derivative of a new field. When this is done, it can be observed that $\mathcal{T}$, m and n form a SU(2)-triplet in the space of the I-indices. With a little bit of redefinition of fields, we obtain our final result that defines the 1D ultra-multiplet. We can dispense with the equations above since their only use was

[4]These were called TM-I and TM-II in reference [4] and are described by equations (1) and (4) there.

to help us derive the final result below defining the 1D ultra-multiplet. Explicitly, its supersymmetry variations take the forms,

$$\begin{aligned}
D_{Ii}\mathcal{G} &= i2C_{ij}C_{IJ}\bar{\varphi}^{Jj} \quad , \\
\bar{D}^{Ii}\mathcal{G} &= 0 \quad , \\
D_{Ii}L_j{}^k &= i2\left[\delta_i{}^k\varphi_{Ij} \;-\; \tfrac{1}{2}\delta_j{}^k\varphi_{Ii}\right] \quad , \\
D_{Ii}R_J{}^K &= 2\left[\delta_I{}^K\varphi_{Ji} \;-\; \tfrac{1}{2}\delta_J{}^K\varphi_{Ii}\right] \quad , \\
D_{Ii}\varphi_{Jj} &= -C_{ij}C_{IJ}(\partial_\tau\bar{\mathcal{G}}) \quad , \\
\bar{D}^{Ii}\varphi_{Jj} &= i(\partial_\tau R_J{}^I)\,\delta_j{}^i \;+\; (\partial_\tau L_j{}^i)\,\delta_J{}^I \quad .
\end{aligned} \tag{2}$$

Thus, the 1D, ultra-multiplet consists of eight bosons $\mathcal{G}$, $L_j{}^k$ and $R_J{}^K$ as well as eight fermions φ_{Ii}.

While in 1D, there is second type of multiplet that can be constructed. To construct it we utilize the 1D automorphic duality transformation once again. This can be done by using the results in (2) as our starting point. First there is simply a "change" the name of the spinor $\varphi_{Ii} \to \zeta_{Ii}$. Next the 1D automorphic duality transformation is implemented by acting only on each scalar field transformation law with ∂_τ. After this step, all scalar fields in the transformation laws appear only through their τ derivatives. These τ derivative terms are then replaced by new independent bosonic fields without τ derivatives. This is also 1D automorphic duality map. Carrying out all of these step, we are led to the "fermionic ultra-multiplet" (FUM) transformation laws,

$$\begin{aligned}
D_{Ii}\mathcal{C} &= i2C_{ij}C_{IJ}\partial_\tau\bar{\zeta}^{Jj} \quad , \\
\bar{D}^{Ii}\mathcal{C} &= 0 \quad , \\
D_{Ii}\mathcal{C}_j{}^k &= i2\left[\delta_i{}^k\partial_\tau\zeta_{Ij} \;-\; \tfrac{1}{2}\delta_j{}^k\partial_\tau\zeta_{Ii}\right] \quad , \\
D_{Ii}\mathcal{C}_J{}^K &= 2\left[\delta_I{}^K\partial_\tau\zeta_{Ji} \;-\; \tfrac{1}{2}\delta_J{}^K\partial_\tau\zeta_{Ii}\right] \quad , \\
D_{Ii}\zeta_{Jj} &= -C_{ij}C_{IJ}\bar{\mathcal{C}} \quad , \\
\bar{D}^{Ii}\zeta_{Jj} &= i\mathcal{C}_J{}^I\,\delta_j{}^i \;+\; \mathcal{C}_j{}^i\,\delta_J{}^I \quad .
\end{aligned} \tag{3}$$

It is a simple matter to show that the supersymmetry variations above uniformly yield a representation of the 1D supersymmetry algebra;

$$\{D_{Ii}\,,\,D_{Jj}\} \;=\; 0 \quad , \quad \{D_{Ii}\,,\,\bar{D}^{Jj}\} \;=\; i2\delta_i{}^j\delta_I{}^J\partial_\tau \quad . \tag{4}$$

This completes the "reduction" procedure of the process. Next we begin the "oxidation" procedure.

Were we to explicitly write out the fermionic derivative in equation (3), we would find that it depends on one bosonic derivative and 8 Grassmannian derivatives as well as their associated coordinates. So the first step of the "oxidation" is to realize that we can consider a transformation that replaces ∂_τ by $\partial_{\pm\!\!\!\!=}$ and thus go up to a 2D heterotic model with (8,0) supersymmetry! In appearance it is almost identical to the equations above with the exception that "+" indices must be appropriately inserted into the equations,

$$
\begin{aligned}
D_{Ii+}\mathcal{G} &= i2C_{ij}C_{IJ}\bar{\varphi}_{+}{}^{Jj} \quad , \\
\bar{D}_{+}^{Ii}\mathcal{G} &= 0 \quad , \\
D_{Ii+}L_j{}^k &= i2\left[\delta_i{}^k\varphi_{Ij+} \;-\; \tfrac{1}{2}\delta_j{}^k\varphi_{Ii+}\right] \quad , \\
D_{Ii+}R_J{}^K &= 2\left[\delta_I{}^K\varphi_{Ji+} \;-\; \tfrac{1}{2}\delta_J{}^K\varphi_{Ii+}\right] \quad , \\
D_{Ii+}\varphi_{Jj+} &= -C_{ij}C_{IJ}(\partial_{\pm\!\!\!\!=}\ \bar{\mathcal{G}}) \quad , \\
\bar{D}^{Ii+}\varphi_{Jj+} &= i(\partial_{\pm\!\!\!\!=}\ R_J{}^I)\ \delta_j{}^i \;+\; (\partial_{\pm\!\!\!\!=}\ L_j{}^i)\ \delta_J{}^I \quad ,
\end{aligned}
\tag{5}
$$

where now we have a realization of the 2D, (8,0) heterotic supersymmetry algebra

$$
\{D_{Ii+}\ ,\ D_{Jj+}\} \;=\; 0 \quad , \quad \{D_{Ii+}\ ,\ \bar{D}_{+}^{Jj}\} \;=\; i2\delta_i{}^j\delta_I{}^J\partial_{\pm\!\!\!\!=} \quad . \tag{6}
$$

The astute reader may well guess what is to follow. The fermionic ultra-multiplet can also be oxidized into a (8,0) representation! As may be guessed from the form of the supersymmetry variations in (3), the FUM naturally oxidizes into an (8,0) "minus spinor" [6] multiplet. We simply need to judiciously introduce "–" indices into (3) as well as make the replacement $\partial_\tau \to \partial_{\pm\!\!\!\!=}$. We find

$$
\begin{aligned}
D_{Ii+}\zeta_{-\ Jj} &= -C_{ij}C_{IJ}\bar{\mathcal{C}} \quad , \\
\bar{D}^{Ii}{}_{+}\zeta_{-\ Jj} &= i\mathcal{F}_J{}^I\ \delta_j{}^i \;+\; \mathcal{F}_j{}^i\ \delta_J{}^I \quad , \\
D_{Ii+}\mathcal{C} &= i2C_{ij}C_{IJ}\partial_{\pm\!\!\!\!=}\ \bar{\zeta}_{-}{}^{Jj} \quad , \\
\bar{D}^{Ii}{}_{+}\mathcal{C} &= 0 \quad , \\
D_{Ii+}\mathcal{F}_j{}^k &= i2\left[\delta_i{}^k(\partial_{\pm\!\!\!\!=}\ \zeta_{-\ Ij}) \;-\; \tfrac{1}{2}\delta_j{}^k(\partial_{\pm\!\!\!\!=}\ \zeta_{-\ Ii})\right] \quad , \\
D_{Ii+}\mathcal{F}_J{}^K &= 2\left[\delta_I{}^K(\partial_{\pm\!\!\!\!=}\ \zeta_{-\ Ji}) \;-\; \tfrac{1}{2}\delta_J{}^K(\partial_{\pm\!\!\!\!=}\ \zeta_{-\ Ii})\right] \quad ,
\end{aligned}
\tag{7}
$$

also provides a realization of the (8,0) heterotic supersymmetry algebra.

In the next section we complete the "oxidation" by obtaining the 2D, N = 8 ultra-multiplet. The careful reader may at this point object, "How can it be that by reducing a 2D, N = 4 model to 1D, performing a 1D automorphic duality map and then oxidizing back, we find a 2D, N = 8 theory?" This almost appears to be magic! It is not quite. The 2D, N = 4 representation from which we started was an off-shell representation. The 2D, N = 8 representation that we find after oxidation is an on-shell theory realizing the supersymmetry algebra,

$$\{ D_{I i \alpha} ~,~ D_{J j \beta} \} ~=~ 0 \quad , \quad \{ D_{I i \alpha} ~,~ \bar D^{J j}{}_{\beta} \} ~=~ i 2 \delta_i{}^j \delta_I{}^J (\gamma^c)_{\alpha\beta} \partial_c \quad . \tag{8}$$

So the original physical plus auxiliary degrees of freedom are converted via reduction, auto-dualization & oxidation into purely physical degrees of freedom afterward. This is the power of 1D automorphic duality! If at a later point, we are able to find the off-shell formulation of the 2D ultra-multiplet, then this process can be repeated to derive an on-shell N = 16 theory. The construction of the off-shell ultra-multiplet will require the "integration" of the fields of an UM together with those of a FUM.

3 The Basic 2D, N = 8 Ultra-multiplet Representation

In the last section, we saw how 2D, N = 8 ultra-multiplets can actually be derived by starting from 2D, N = 4 hypermultiplets. Here we start by giving the simplest ultra-multiplet action

$$\begin{aligned} {\cal L}_{\rm UM} ~=~ [~ & \tfrac{1}{2} (\partial^a \bar{\cal G})(\partial_a {\cal G}) ~+~ \tfrac{1}{4} (\partial^a L_j{}^k)(\partial_a L_k{}^j) ~+ \\ & \tfrac{1}{4} (\partial^a R_J{}^K)(\partial_a R_K{}^J) ~-~ i \bar\varphi^{I i \alpha} (\gamma^c)_{\alpha\beta} \partial_c \varphi^{\beta}{}_{I i} ~] \quad , \end{aligned} \tag{9}$$

which is left invariant under the 2D, N = 8 supersymmetry variations given by

$$\begin{aligned} D_{I i \alpha} {\cal G} ~&=~ i 2 C_{ij} C_{IJ} \bar\varphi_{\alpha}{}^{J j} \quad , \\ \bar D^{I i}_{\alpha} {\cal G} ~&=~ 0 \quad , \\ D_{I i \alpha} L_j{}^k ~&=~ i 2 \left[\delta_i{}^k \varphi_{I j \alpha} ~-~ \tfrac{1}{2} \delta_j{}^k \varphi_{I i \alpha} \right] \quad , \\ D_{I i \alpha} R_J{}^K ~&=~ 2 \left[\delta_I{}^K \varphi_{J i \alpha} ~-~ \tfrac{1}{2} \delta_J{}^K \varphi_{I i \alpha} \right] \quad , \\ D_{I i \alpha} \varphi_{J j \beta} ~&=~ - C_{ij} C_{IJ} (\gamma^c)_{\alpha\beta} (\partial_c \bar{\cal G}) \quad , \end{aligned}$$

$$\bar{D}^{I i\alpha}\varphi_{Jj\beta} \;=\; i(\gamma^c)_{\alpha\beta}(\partial_c R_J{}^I)\;\delta_j{}^i \;+\; (\gamma^c)_{\alpha\beta}(\partial_c L_j{}^i)\;\delta_J{}^I \quad . \tag{10}$$

One of the most interesting features of the ultra-multiplet is the group of automorphism that it realizes on the 8 supersymmetry generators. The group turns out to be $SU(2)\otimes SU(2)\otimes U(1)$. This non-semisimple group is much smaller than the expected $SO(8)$ normally assumed to appear in a 2D, N = 8 superconformal theory. As can be seen from the action, this theory is clearly scale invariant. In fact, the existence of the UM and FUM theories, suggests the existence of an (8,0) (as well as N = 8) 2D supergravity multiplet with only seven gauge fields gauging $SU(2)\otimes SU(2)\otimes U(1)$.

4 Parity Twists of the Ultra-Multiplet Theory

Sometime ago, the concepts of variant representations [7] and twisted multiplets [8] were introduced introduced. These are useful to recall, because they allow us to use the basic ultra-multiplet to derive additional representations of the N = 8 supersymmetry. The use of a parity twist is a useful way to find these. The idea is simple. Given a representation of 2D supersymmetry, it is possible to find a new and distinct representation by replacing scalar spin-0 fields by pseudo-scalar spin-0 fields. Thus, there are variants ultra-multiplets that contain one, two, three and four pseudo-scalars (any more than this is equivalent to one of these cases). We will refer to these as the twisted ultra-multiplets I thru IV (i.e. TUM-I, TUM-II, TUM-III and TUM-IV).

We begin our discussion by considering the TUM-I theory. The parity twist is incorporated into this model by defining its supersymmetry variations as,

$$\begin{aligned}
D_{Ii\alpha}\tilde{\mathcal{A}} \;&=\; i2C_{ij}C_{IJ}\bar{\rho}_\alpha{}^{Jj} \quad , \\
D_{Ii\alpha}\tilde{\mathcal{B}} \;&=\; -2C_{ij}C_{IJ}(\gamma^3)_\alpha{}^\delta\bar{\rho}_\delta{}^{Jj} \quad , \\
D_{Ii\alpha}\tilde{L}_j{}^k \;&=\; i2\left[\delta_i{}^k\rho_{Ij\alpha} \;-\; \tfrac{1}{2}\delta_j{}^k\rho_{Ii\alpha}\right] \quad , \\
D_{Ii\alpha}\tilde{R}_J{}^K \;&=\; 2\left[\delta_I{}^K\rho_{Ji\alpha} \;-\; \tfrac{1}{2}\delta_J{}^K\rho_{Ii\alpha}\right] \quad , \\
D_{Ii\alpha}\rho_{Jj\beta} \;&=\; -C_{ij}C_{IJ}[\;(\gamma^c)_{\alpha\beta}(\partial_c\tilde{\mathcal{A}}) \;-\; i(\gamma^3\gamma^c)_{\alpha\beta}(\partial_c\tilde{\mathcal{B}})\;] \quad , \\
\bar{D}^{Ii\alpha}\rho_{Jj\beta} \;&=\; i(\gamma^c)_{\alpha\beta}(\partial_c\tilde{R}_J{}^I)\;\delta_j{}^i \;+\; (\gamma^c)_{\alpha\beta}(\partial_c\tilde{L}_j{}^i)\;\delta_J{}^I \quad .
\end{aligned} \tag{11}$$

Above the component field $\mathcal{B}$ is the pseudo-scalar that replaces a scalar in the basic ultra-multiplet. The action for the multiplet is exactly the same (in form) as that for the basic ultra-multiplet. One of the most interesting aspects of the TUM-I model is

that it is related to a 2D, N = 8 vector multiplet. This is most clearly seen by writing the commutator algebra for the 2D, N = 8 gauge U(1) covariant derivative.

$$
\begin{aligned}
[\ \nabla_{iI\alpha}\ ,\ \nabla_{jJ\beta}\ \} \ &= \ -4gC_{\alpha\beta}[\ C_{IJ}\tilde{L}_i{}^kC_{kj}\ -\ iC_{ij}\tilde{R}_I{}^KC_{KJ}\ ] \quad , \\
[\ \nabla_{iI\alpha}\ ,\ \bar{\nabla}^{jJ}{}_\beta\ \} \ &= \ i2\delta_i{}^j\delta_I{}^J(\gamma^c)_{\alpha\beta}\nabla_c\ +\ 2g\delta_i{}^j\delta_I{}^J[\ C_{\alpha\beta}\tilde{\mathcal{A}}\ +\ i(\gamma^3)_{\alpha\beta}\tilde{\mathcal{B}}\ ] \quad , \\
[\ \nabla_{iI\alpha}\ ,\ \nabla_b\ \} \ &= \ g(\gamma^c)_\alpha{}^\gamma\bar{W}_{iI\gamma} \quad , \\
[\ \nabla_a\ ,\ \nabla_b\ \} \ &= \ -ig\epsilon_{ab}\mathcal{W} \quad .
\end{aligned} \tag{12}
$$

The Bianchi Identities that follow from these equations have a solution that is closely related to those in (9.)[5]. We need only identify $\bar{W}_{iI\alpha} = -\frac{1}{2}C_{ij}C_{IJ}\bar{\rho}^{jJ}{}_\alpha$ and to slightly modify one of our previous results to,

$$
D_{I i\alpha}\rho_{Jj\beta} \ = \ C_{ij}C_{IJ}(\gamma^3)_{\alpha\beta}\mathcal{W} - C_{ij}C_{IJ}[\ (\gamma^c)_{\alpha\beta}(\partial_c\tilde{\mathcal{A}})\ -\ i(\gamma^3\gamma^c)_{\alpha\beta}(\partial_c\tilde{\mathcal{B}})\ ] \quad . \tag{13}
$$

This result is the usual one that follows in a 2D supersymmetric theory when one compares a scalar multiplet to a vector multiplet. (In an off-shell formulation of the TUM-I model, $\mathcal{W}$ is replaced by an auxiliary field. This is the beginning of the off-shell formulation of the TUM-I theory.) A final point of interest regarding this form of the ultra-multiplet is that this version can be "oxidized" all the way back to 4D where it can be recognized as the 4D, N = 4 Yang-Mills theory.

The next ultra-multiplet is the TUM-II theory which possesses two pseudo-scalars among its fields. Its supersymmetry variations are given by

$$
\begin{aligned}
D_{Ii\alpha}\mathcal{H} \ &= \ i2C_{ij}C_{IJ}(\gamma^3)_\alpha{}^\beta\bar{\lambda}_\beta{}^{Jj} \quad , \\
\bar{D}^{Ii}_\alpha\mathcal{H} \ &= \ 0 \quad , \\
D_{Ii\alpha}\mathcal{B}_j{}^k \ &= \ i2\left[\delta_i{}^k\lambda_{Ij\alpha}\ -\ \tfrac{1}{2}\delta_j{}^k\lambda_{Ii\alpha}\right] \quad , \\
D_{Ii\alpha}\mathcal{A}_J{}^K \ &= \ 2\left[\delta_I{}^K\lambda_{Ji\alpha}\ -\ \tfrac{1}{2}\delta_J{}^K\lambda_{Ii\alpha}\right] \quad , \\
D_{Ii\alpha}\lambda_{Jj\beta} \ &= \ -C_{ij}C_{IJ}(\gamma^3\gamma^c)_{\alpha\beta}(\partial_c\bar{\mathcal{H}}) \quad , \\
\bar{D}^{Ii\alpha}\lambda_{Jj\beta} \ &= \ i(\gamma^c)_{\alpha\beta}(\partial_c\mathcal{A}_J{}^I)\ \delta_j{}^i\ +\ (\gamma^c)_{\alpha\beta}(\partial_c\mathcal{B}_j{}^i)\ \delta_J{}^I \quad .
\end{aligned} \tag{14}
$$

Continuing along the same lines, there is the TUM-III theory containing three pseudo-scalars in its spectrum. Here the supersymmetry variations are defined by,

$$
D_{Ii\alpha}\mathcal{M} \ = \ i2C_{ij}C_{IJ}\bar{\Phi}_\alpha{}^{Jj} \quad ,
$$

[5]It can be observed that these results may be derived by applying simple dimensional compactification to 4D, N = 4 Yang-Mills theory. This provides an interesting and independent confirmation of the existence of the ultra-multiplet.

$$
\begin{aligned}
\bar{D}_\alpha^{Ii}\mathcal{M} &= 0 \quad , \\
D_{Ii\alpha}X_j{}^k &= i2\left[\delta_i{}^k\Phi_{Ij\alpha} \;-\; \tfrac{1}{2}\delta_j{}^k\Phi_{Ii\alpha}\right] \quad , \\
D_{Ii\alpha}Y_J{}^K &= 2(\gamma^3)_\alpha{}^\beta\left[\delta_I{}^K\Phi_{Ji\beta} \;-\; \tfrac{1}{2}\delta_J{}^K\Phi_{Ii\beta}\right] \quad , \\
D_{Ii\alpha}\Phi_{Jj\beta} &= -C_{ij}C_{IJ}(\gamma^c)_{\alpha\beta}(\partial_c\bar{\mathcal{M}}) \quad , \\
\bar{D}^{Ii\alpha}\Phi_{Jj\beta} &= i(\gamma^3\gamma^c)_{\alpha\beta}(\partial_c Y_J{}^I)\;\delta_j{}^i \;+\; (\gamma^c)_{\alpha\beta}(\partial_c X_j{}^i)\;\delta_J{}^I \quad .
\end{aligned} \tag{15}
$$

Finally, there is the TUM-IV theory containing four pseudo-scalars in its spectrum. Analogous to the previous versions of this theory, the supersymmetry variations are given by,

$$
\begin{aligned}
D_{Ii\alpha}\tilde{\mathcal{M}} &= i2C_{ij}C_{IJ}\bar{\Lambda}_\alpha{}^{Jj} \quad , \\
D_{Ii\alpha}\tilde{\mathcal{N}} &= -2C_{ij}C_{IJ}(\gamma^3)_\alpha{}^\delta\bar{\Lambda}_\delta{}^{Jj} \quad , \\
D_{Ii\alpha}U_j{}^k &= i2(\gamma^3)_\alpha{}^\delta\left[\delta_i{}^k\Lambda_{Ij\delta} \;-\; \tfrac{1}{2}\delta_j{}^k\Lambda_{Ii\delta}\right] \quad , \\
D_{Ii\alpha}V_J{}^K &= 2\left[\delta_I{}^K\Lambda_{Ji\alpha} \;-\; \tfrac{1}{2}\delta_J{}^K\Lambda_{Ii\alpha}\right] \quad , \\
D_{Ii\alpha}\Lambda_{Jj\beta} &= -C_{ij}C_{IJ}[\;(\gamma^c)_{\alpha\beta}(\partial_c\tilde{\mathcal{M}}) \;-\; i(\gamma^3\gamma^c)_{\alpha\beta}(\partial_c\tilde{\mathcal{N}})\;] \quad , \\
\bar{D}^{Ii\alpha}\Lambda_{Jj\beta} &= i(\gamma^c)_{\alpha\beta}(\partial_c V_J{}^I)\;\delta_j{}^i \;+\; (\gamma^3\gamma^c)_{\alpha\beta}(\partial_c U_j{}^i)\;\delta_J{}^I \quad .
\end{aligned} \tag{16}
$$

The form of the action for all of the ultra-multiplets is given by equation (7).

5 SU(2) $\otimes$ SO(2) Ultra-Multiplets

The starting point of our discussions was the reduction, dualization and oxidation of the THM-II, N = 4 theory. However, we also could have used the THM-I, N = 4 theory as the starting point! Carrying out the reduction leads to the intermediate results,

$$
\begin{aligned}
D_{Ii}F &= 2C_{ij}\lambda_I{}^j \quad , \\
\bar{D}_I{}^iF &= 0 \quad , \\
D_{Ii}S &= -i\bar{\lambda}_{Ii} \quad , \\
D_{Ii}P &= (\sigma^3)_I{}^J\bar{\lambda}_{Ji} \quad , \\
D_{Ii}\lambda^{Jj} &= \delta_i{}^j\,[\;\delta_{IJ}(\partial_\tau S) \;+\; i(\sigma^3)_{IJ}(\partial_\tau P)\;] \\
&\quad -i\;\left[\;\tfrac{1}{2}\delta_i{}^j(\sigma^1)_{IJ}(\partial_\tau\varphi) \;-\; 2(\sigma^2)_{IJ}(\partial_\tau\varphi_i{}^j)\;\right] \quad , \\
\bar{D}_I{}^i\lambda_J{}^j &= iC^{ij}\delta_{IJ}(\partial_\tau F) \quad , \\
D_{\alpha i}\varphi &= -2(\sigma^1)_I{}^J\bar{\lambda}_{Ji} \quad ,
\end{aligned}
$$

$$D_{Ii}\varphi_j{}^k = \quad - (\delta_j{}^l \delta_i{}^k - \tfrac{1}{2}\delta_j{}^k \delta_i{}^l)(\sigma^2)_I{}^J \bar{\lambda}_{Jl} \quad , \tag{17}$$

As can be seen this 1D theory only has only $SO(2) \otimes SU(2) \otimes U(1)$ symmetry. Its oxidation back to 2D retains this structure. This is the beginning of a whole set of similar such theories. But all of these theories are related by a redefinition to the previous discussed theories. In particular, one need only perform the redefinition $D_{I i\alpha} \to (\sigma^2)_I{}^J D_{J i\alpha}$.

6 Summary and Conclusion

We have seen that rigid 2D, N = 8 representations are very abundant. The ultra-multiplet, in all of its guises, manifest a very small group of automorphisms on the supersymmetry derivatives (typically only $SU(2) \otimes SU(2) \otimes U(1)$ or $SO(2) \otimes SU(2) \otimes U(1)$. It is trivially the case that the foremost of these can also be regarded as $SO(4) \otimes U(1)$ or $SO(4) \otimes SO(2)$ groups. All of these are much smaller than the expected $SO(8)$ of the known 2D, N = 8 superconformal theories. Furthermore the rigid actions for all of these models are scale invariant. The real remaining challenge is to find out whether there exist 2D, N = 8 conformal supergravity theories that can be coupled to ultra-multiplets. None of the standard constructions associated with conformal supergroups seem compatible with ultra-multiplets! It is just possible that presently unknown N = 8 string-like theories may be waiting to be discovered.

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Appendix: $SO(4) \otimes SO(2)$ Formulation of the Ultra-multiplet

In this appendix, we wish to give an alternate formulation of the ultra-multiplet. We wish to take advantage of the fact that $SU(2) \otimes SU(2) \otimes U(1)$ is equivalent to $SO(4) \otimes U(1)$ This implies that the spinor derivative may be given in the form $D_{i\alpha}$ here the i-index takes on four values (i.e. vector indices in $SO(4)$). These derivatives are still complex so a rigid phase rotation may act to realize a U(1) (SO(2)) symmetry upon them. The component fields of the basic ultra-multiplet can be expressed as $\mathcal{G}$, $\varphi_{i\alpha}$, $L_{\hat{a}}$ and $R_{\hat{a}}$. The supersymmetry variations of the 1D theory take the form,

$$
\begin{aligned}
D_i \mathcal{G} &= i2\delta_{ij}\bar{\varphi}^j \quad , \\
\bar{D}^i \mathcal{G} &= 0 \quad , \\
D_i L_{\hat{a}} &= i2(\alpha_{\hat{a}})_i{}^j \varphi_j \quad , \\
D_i R_{\hat{a}} &= 2(\beta_{\hat{a}})_i{}^j \varphi_j \quad , \\
D_i \varphi_j &= -\delta_{ij}(\partial_\tau \bar{\mathcal{G}}) \quad , \\
\bar{D}^i \varphi_j &= (\alpha^{\hat{a}})_j{}^i (\partial_\tau L_{\hat{a}}) \; + \; i(\beta^{\hat{a}})_j{}^i (\partial_\tau R_{\hat{a}}) \qquad .
\end{aligned}
\tag{A.1}
$$

In these expressions, the quantities $(\alpha^{\hat{a}})_i{}^j$ and $(\beta^{\hat{a}})_i{}^j$ represent two commuting sets of real, four by four, antisymmetric $SU(2)$ matrices. Taken together these six matrices represent the generators of $SO(4)$. These quantities are well known in the physics literature [9]. The Lagrangian can be written concisely as

$$
\mathcal{L}_{\rm UM} \; = \; [\; \tfrac{1}{4}(\partial_\tau \bar{S}_j{}^k)(\partial_\tau S_k{}^j) \; - \; i\bar{\varphi}^i \partial_\tau \varphi_i \;] \quad , \tag{A.2}
$$

where $S_i{}^j \equiv \; \delta_i{}^j \mathcal{G} \; + \; (\alpha^{\hat{a}})_i{}^j L_{\hat{a}} \; + \; i(\beta^{\hat{a}})_i{}^j R_{\hat{a}}$.

References

[1] M. B. Green and J. H. Schwarz, Phys. Lett. **136B** (1984) 367; idem. Nucl. Phys. **B198** (1982) 252, 441.

[2] D. Gross, J. A. Harvey, E. Martinec and R. Rohm, Phys. Rev. Lett. **54** (1985) 502; Nucl. Phys.**B256** (1986) 253; ibid. **B267** (1986) 75 .

[3] S. J. Gates, Jr., L. Lu and R. Oerter, Phys. Lett. **218B** (1989) 33.

[4] S. J. Gates, Jr., "Why Are There So Many N = 4 Superstrings?", Univ. of Md. Preprint # UMDPP 95-06, (July, 1994) to appear in Phys. Lett. B.

[5] S. J. Gates, Jr. and L. Rana, "On Extended Supersymmetric Quantum Mechanics", Univ.of Md. Preprint # UMDPP 93-194, (September, 1994).

[6] C. M. Hull and E. Witten, Phys. Lett. **160B** (1985) 398; R. Brooks, S. J. Gates, Jr. and F. Muhammad, Nucl. Phys. **B268** (1986) 599.

[7] S. J. Gates, Jr. and W. Siegel, Nucl. Phys. **B187** (1981) 389.

[8] S. J. Gates, Jr., C. M. Hull and M. Roček, Nucl. Phys. **B248** (1984) 157.

[9] G. t'Hooft, Phys. Rev. **D14** (1976) 3432; J. Scherk, *Extended Supersymmetry and Extended Supergravity Theory*, Cargése Lecture Notes, LPTENS preprint 78/21 (September, 1978).